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Numerical Modeling of Fretting Fatigue Crack Propagation based on a Combined XFEM and Mortar Contact Approach

Abbas Moradi, Saeed Adibnazari, Mohsen Safajuy

Abstract— In this Paper, crack propagation in the fretting fatigue problems has been investigated. In order to include the contact constraints, the mortar formulation is employed. Mortar method provides a powerful algorithm for solving frictional contact problems. For modeling of cracks and crack growth, the eXtended Finite Element Method (XFEM) due to its influential capability for modeling of cracks and crack growth, is employed. In this method, the finite element mesh does not need to conform to the crack. Therefore, only a single and regular mesh can be used for any crack geometry. Classical law of coulomb is applied to model the friction. The consequential sets of non-linear equations are solved using an efficient numerical algorithm based on Updated Lagrangian formulation and Modified Newton–Raphson iteration method. A two-dimensional computer code is developed based on proposed procedures, and crack propagation in fretting fatigue problems is investigated. On utilizing the non-linear contact capabilities of the code, the numerical technique is applied to solve some typical problems of contact. Then a fretting fatigue problem is modeled and the results are validated by comparison with the results of fretting fatigue tests. The presented numerical examples show that the presented formulation exhibits a suitable accuracy and has a reasonable convergence rate.

Index Terms— Fretting Fatigue, Frictional contact, Mortar finite element method, XFEM, Crack Propagation

1 Introduction

There are many significant contact fatigue problems in structural design which fretting fatigue at lap joints in aging aircraft and fretting fatigue of dovetail joints in gas turbine engine are the most important of them. In spite of considerable successes in this area during the past two decades, there are still a lot of problems, and many researchers are trying to solve them. To avoid comparatively expensive experimental full-scale tests, numerical methods can be used to simulate fretting problems.

Fretting is usually the result of very small amplitude oscillatory relative motion between two contacting parts. A crack can impulsively nucleate because of fretting when compared to crack nucleation due to pure fatigue. Because fretting considerably reduces the time to crack initiation, the largest part of the fretting fatigue life consists of fatigue crack propagation. Adibnazari and Hoeppner [1], 1994, showed that in fretting fatigue, most of the component life is spent in the propagation stage.

Many parameters such as contact geometry, contact load, friction and material strength can influence the resistance of materials against fretting damage. Experimental study of fretting fatigue is very difficult, expensive and not very accurate. Numerical simulations provide an appropriate and powerful tool for parametric study of fretting fatigue behavior of materials and crack growth phenomena.

- Ph.D. Student, Department of Aerospace Engineering, Sharif University of Technology, Tehran, Iran, P.O. Box: 11365-11155, Phone: +98-21-66164601, Fax: +98-21-66022731. E-mail: amoradi@ae.sharif.edu
- Professor, Department of Aerospace Engineering, Sharif University of Technology, Tehran, Iran, P.O. Box: 11365-11155, Phone: +98-21-66164601, Fax: +98-21-66022731. E-mail: adib@sharif.edu
- Masters of Engineering, Department of Aerospace Engineering, Sharif University of Technology, Tehran, Iran. E-mail: safajuy@alum.sharif.edu

In this study, only the crack propagation stage is considered and an initial crack with proper length is assumed. In Linear fracture mechanic, calculation of correct values of the stress intensity factors (SIFs) is very important and necessary for the crack propagation modeling. Evaluating stress intensity factor in the problems including the crack-contact interaction is complicated. Because it is a function of the position along the crack front, crack size and shape, geometry of the structure and contact constraints. In this work, the XFEM were used to perform a linear fracture mechanics analysis of two-dimensional contact problems.

Recently, researchers proposed XFEM to model discontinuity fields particularly in cracked structures [2], [3], [4], [5]. In this method, finite element formulation enriched using additional functions to model discontinuity in displacement and strain fields. Thus, applying XFEM leads us to a proper solution for problems in cracked structures, without a need for very fine meshes. Via this method, the crack's boundary conditions are transformed appropriate to the nodes. For this objective, degree of freedom of those nodes located in close proximity to the crack is virtually increased.

There are also many researchers trying to apply XFEM for solving fretting contact problems. Giner et al. [4] applied the XFEM for the analysis of fretting fatigue problems by using a two-dimensional implementation of the XFEM within the finite element software ABAQUS. Baetto et al. [6] used the Large Time Increment method (LATIN) in order to solve the non-linear XFEM contact with friction formulation. Baetto et al. [7] used an XFEM frictional fatigue crack model to compute stress intensity factors and presented a combined experimental and numerical study in order to predict fretting crack propagation. The employ of the XFEM allows the assessment of crack propagation in a very efficient way. The J-integral methodology is used to estimate accurate values of the SIF along the crack. Using a standard finite element formulation would be far less accurate and extremely costly, because of the need for mesh regeneration at each time.

In recent years, research for the segment-to-segment discretization strategies became very active to capture contact nonlinearity. At first, Simo et al. [8], Papadopoulos and Taylor [9], Zavarise and Wriggers [10] applied this approach for geometrical linear problems. A large amount of the recent segment-to-segment approaches are based on the mortar finite element method. A distinguishing characteristic of the mortar method is the imposition of interfacial constraints, such as non-penetration conditions in the case of frictionless contact. This method has many advantages when compared with classical node to-segment contact formulations [11], [12]. In contrast to the node-to-segment discretization, the continuity constraints are not enforced at discrete finite element nodes but are formulated along the complete coupling boundary in a weak integral sense.

In this paper, the influence of the contact stress field on the crack growth path is studied. To this end, a small primary crack located near the contact zone is assumed, and crack propagation is analyzed. The contact between crack faces is not modelled in this work.

Updated Lagrangian formulation and Modified Newton–Raphson iteration method used to solve sets of non-linear equations. A two-dimensional computer code is provided to solve some typical problems of contact and fretting fatigue. The results are validated by comparison with the results of some fretting fatigue tests and those presented in the literature. The presented numerical examples show that the presented formulation has a suitable accuracy and exhibits a good convergence rate.

2 XFEM Formulation

In XFEM, finite element formulation enriched using additional functions to model discontinuity in displacement and strain fields. Thus, applying XFEM leads us to a proper solution for problems in cracked structures, without a need for very fine meshes. Via this method, the crack's boundary conditions are transformed appropriate to the nodes. For this objective, degree of freedom of those nodes located in close proximity to the crack is virtually increased.

2.1 Cracks Modeling using Enrichment Functions

Application of the level set functions to distinguish the location of discontinuities and enrich nodes near elements including cracks, is presented in this section. The presented approach consists of a standard finite element model and a crack representation, which is independent of the elements. This is achieved by adding degree of freedom to those nodes near the crack.

Depending on whether the crack completely crosses the element or not, the interpolation functions of the element are varied. For the case in which the crack crosses the element, the modified Heaviside function is exercised. For another case, when the crack tip is in the element, the displacement field is enriched using a singular function.

2.2 Level Set Function

Consider the boundary $\Omega \subseteq \mathbb{R}^2$ that contains an internal boundary Γ that represents an arbitrary curved crack as depicted in fig. 1. In order to develop the XFEM formulation to solve this curved crack, a level set framework is adopted by representing the crack as the zero level set of the function below:

$$\varphi(x) = min\|x - x_\Gamma\| sign(\vec{n}.(x - x_\Gamma)) \tag{1}$$

$$\vec{n} = \frac{\nabla F}{\|\nabla F\|} \tag{2}$$

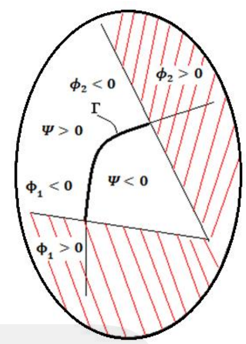

Fig. 1. Presentation of the crack as the zero level set of a function Ψ(x)

An endpoint of the crack is represented as the intersection of the zero level set of ψ with an orthogonal zero level set of the function $\phi_i(x)$, where i shows the crack's tip number as fig. 1. Throughout the solution procedure, the crack will be discretized to a number of points and then the necessary values of the level set functions are calculated and stored for these points and material nodes around the crack.

The level set functions that represent the crack tip are initially defined by $\phi_i(x) = (x - x_i) * \bar{t}$, where x_i is the location of the i^{th} crack tips. Also $\bar{t}$ is a unit vector tangent at this location.

2.3 Node Enrichment

In XFEM, the mesh is completely independent of the location and geometry of the crack. The discontinuities across the crack are modeled by enrichment functions. Consider the XFEM displacement approximation for a vector valued function $u(x): R^2 \rightarrow R^2$ given following equation:

$$u(x) = \sum_{i \in I} u_i \emptyset_I(x) + \sum_{j \in J} a_j \emptyset_j(x) H(\Psi(x)) + \sum_{k \in K} \emptyset_k(x) \left(\sum_{l=1}^{4} b_k^l F_l(r, \theta) \right) \tag{3}$$

Where $\emptyset_I(x)$ is the shape function associated with node I, that I is set of all nodes. In addition, J is the set of those nodes that correspond to the cut elements by crack. The set k contains the nodes of those elements containing the crack tips. u_I, a_J and b_k are also the nodal degrees of freedom corresponding to the

displacement.
The second important and distinguishing factor to note in (3) is the enrichment functions $H(x)$ and $F_l(r,\theta)$. The function H is defined as follows:

$$H(x) = \begin{cases} +1\ ; & if\ \ (x - x_\Gamma) \geq 0 \\ -1\ ; & if\ \ (x - x_\Gamma).n < 0 \end{cases} \quad (4)$$

Where z=zero is defined to be along the crack. This implies that the discontinuity occurs at the location of the crack. For one-material structures, the branch function F_l is defined by following equation:

$$F_l(r,\theta) = \{\sqrt{r}\,sin(\theta/2)\ ,\sqrt{r}\,sin(\theta/2)\ sin(\theta)\ ,$$
$$\sqrt{r}\ \ sin(\theta)\ \ cos(\theta/2)\ \} \quad (5)$$

2.4 Driving of Stiffness Matrix

In the finite element method, stiffness matrix of an element can be derived using elastic strain energy equation as follows:

$$k_e = \int_{V^e} [B^e]^T\,[D][B^e]dV^e \quad (6)$$

Where $[D]$ is the stress-strain matrix,$[B^e]$ is the strain-displacement relationship matrix. For two-dimensional solid structures, the strain-displacement relationships are considered as follows:

$$[\varepsilon] = \begin{bmatrix} \dfrac{\partial u_X}{\partial x} \\ \dfrac{\partial u_y}{\partial y} \\ \dfrac{\partial u_X}{\partial x} + \dfrac{\partial u_y}{\partial y} \end{bmatrix} \quad (7)$$

That [ε] is the strain of an element, [q] is the displacement due to degree of freedom at the element nodes. In addition, the strain-displacement relationship matrix for XFEM equation can be decomposed as follow:

$$[B^e] = [B_i^u \quad B_i^a B_i^b B_i^c] \quad (8)$$

That $[B_i^u]$, $[B_i^a]$, $[B_i^b]$ and $[B_i^c]$ refer to sets of i, j, k, r, respectively.

$$K^e = \int_{\Omega^e} B^{eT}\,C^e B^e d_{\Omega^e} =$$

$$\int_{-1}^{1}\int_{-1}^{1}\int_{\Omega^e} B^{eT}(\xi,\eta)\,C^e B^e\,(\xi,\eta)\,det(J)\,d_\xi d_\eta$$
(9)

Now, as a traditional finite element, assembly of stiffness matrix and load vector can determine nodal displacements. Afterwards, by using displacement of crack tip, the stress intensity factor can be computed.

2.5 Calculation of Stress intensity factors (SIFs)

There are several methods to calculate stress intensity factor, such as path-independent interaction integral and Green function. One of the popular methods is the J-integral method. The J-integral method has good accuracy and little user intervention for the SIF computation. In addition, this method enables the extraction of K_I and K_{II} for mixed-mode problems by using auxiliary fields [13].

$$I^{(1,2)} = \int_\Gamma \left[W^{(1,2)}\delta_{1j} - \sigma_{ij}^{(1)}\frac{\partial u_i^{(2)}}{\partial x_i} - \sigma_{ij}^{(2)}\frac{\partial u_i^{(1)}}{\partial x_i} \right] n_j\ d\Gamma \quad (10)$$

$$W^{(1,2)} = \sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)} = \sigma_{ij}^{(2)}\varepsilon_{ij}^{(1)} \quad (11)$$

$$I^{(1,2)} = \frac{2\left(K_I^{(1)}K_I^{(2)} + K_{II}^{(1)}K_{II}^{(2)}\right)}{E_{eff}} \quad (12)$$

Where $^{(1)}$ are the actual fields of the problem approximated by the XFEM solution and $^{(2)}$ are the auxiliary fields. These fields are chosen to be the asymptotic crack tip fields for pure mode I or pure mode II to compute K_I and K_{II}, respectively. In (10), x_i are the local directions with respect to the crack tip, δ_{1j}j is the Kronecker's delta. The SIFs of the problem are then calculated as follows:

$$K_I^{(1)} = I^{(1,Mode\ I)}E_{eff}/2 \quad (13)$$

$$K_{II}^{(1)} = I^{(1,Mode\ II)}E_{eff}/2 \quad (14)$$

3 Contact Formulation

Contact condition for normal direction or so called Kuhn-Tucker-Karush condition, which ensures non-penetration constraint and prevents non-adhesive traction and has introduced as following equations:

$$g_N \geq 0, \qquad p_N \leq 0,$$
$$p_N g_N = 0 \quad (15)$$

Where g_N and p_N are gap and pressure functions, respectively. Definition of a contact potential is a very common method, as already presented in [14] and applied by Fischer and Wriggers [15]:

$$minimize\!:\ \Pi_T\ (x) = \Pi_{int}\ (x) + \Pi_{ext}\ (x) + \Pi_c$$
$$subject\ to\!:\quad g_N(x) \geq 0 \quad (16)$$

$$\Pi_c = \int g_c p_c\ d\Gamma \quad (17)$$

$$\delta\Pi_T = \delta\Pi_{int} + \delta\Pi_{ext} + \delta\Pi_c = 0 \quad (18)$$

$$\delta\Pi_c = \int(\delta g_c\ p_c + g_c\ \delta p_c)\ d\Gamma \geq 0 \quad (19)$$

Total potential energy of contact system showed in [15], should be minimized subject to the contact conditions.
For frictional contact, Coulomb formulation is applied.

$$t_T = -\mu|p_N|\,\frac{\dot{g}_T}{\|\dot{g}_T\|} \qquad if\ \ \|t_T\| > \mu|p_N| \quad (20)$$

Where g_T and μ are tangential gap and friction coefficient, respectively.

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3.1 Mortar Method

The mortar method is a special technique to implement contact constraints in the discretized system for non-matching meshes. The method is based on a Lagrange multiplier formulation in which special interpolation functions are used to discretize the Lagrange multiplier in the contact interface. In Lagrange Multiplier method, the potential energy of contact are considered as follows [16]:

$$\Pi_c^{LM} = \int (\lambda_N g_N + \lambda_T g_T) d\Gamma \quad (21)$$

Where λ_N and λ_T are Lagrange multipliers. After variation, contact term is equal to following equations:

$$C_c^{LM} = \int_{\Gamma_c} (\lambda_N \delta g_N + \lambda_T \delta g_T) d\Gamma + \int_{\Gamma_c} (\delta\lambda_N g_N + \delta\lambda_T g_T) d\Gamma \quad (22)$$

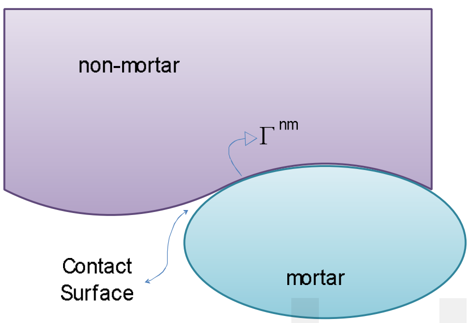

FIG. 2. MORTAR AND NON-MORTAR CONTACT SURFACE

$$\int_{\Gamma_c} (\delta\lambda_N g_N) d\Gamma = \sum_{c=1}^{n_c} \int_{\Gamma_c^{nm}} \left(\delta\lambda_N(\zeta) g_N(\zeta,\xi)\right) d\Gamma$$

$$= \sum_{c=1}^{n_c} \sum_{K} \delta\lambda_{N_K} \int_{\Gamma_c^{nm}} M_K(\zeta) \left(\sum_{I} N_I^{nm}(\zeta)\, u_s^{nm} - \sum_{J} N_J^{m}(\xi)\, u_m^{m} \right) \cdot n^m(\xi)\, d\Gamma = 0 \quad (23)$$

$$g_N = [u_s^{nm}(\zeta) - u_m^{m}(\xi)] \cdot n^m(\xi) \quad (24)$$

$$u_s^{nm}(\zeta) = \sum_{i=1}^{N} N_i^{nm}(\zeta) u_{si}^{nm} \quad (25)$$

$$X_s^{nm}(\zeta) = \sum_{i=1}^{N} N_i^{nm}(\zeta) X_{si}^{nm} \quad (26)$$

$$\lambda_{N_s}(\zeta) = \sum_{k=1}^{N} M_k^{nm}(\zeta) \lambda_{Ns_k} \quad (27)$$

3.2 Fretting Fatigue

In order to apply a Newton–Raphson's method to solve the non-linear system of equations, the contact virtual work terms are linearized. The algorithms proposed for solving of a fretting fatigue problem shown in fig. 3 and 4.

There is a loop to find the correct Stress Intensity Factor using XFEM Method. Another loop is proposed to find the correct contact conditions for every node (if the node is or not in contact) and solve the non-linear equation using Newton–Raphson's method.

- Start
- Loop (Crack Growth Increment)
- Loop (Contact Iteration)
- Use XFEM Module
- Run Level Set Module
- Find Enrichment Nodes/Elements
- Loop On Elements
- Loop On Gauss Points
- Calculate Stiffness Matrix

FIG. 3. ALGORITHM FOR FINDING STRESS INTENSITY FACTOR USING XFEM METHOD

In second loop, the beginning step consists of computing the average gap function for every non-mortar nodes and nodes with negative gap are activated.

- Start
- Input Contact Elements/Nodes
- Loop On Contact Gauss Points
- Calculate Gap Function
- If Gap <= Tolerance
- Active GP/Nodes/Segment
- Loop On Active Gauss Points
- Calculate Contact Residual
- Assemble Contact Residual
- Assemble Global Stiffness
- Apply B.C. & External Load
- Solve Nonlinear Equations

FIG. 4. ALGORITHM FOR FINDING CONTACT CONDITIONS FOR EVERY NODE

4 NUMERICAL EXAMPLES

The proposed algorithm derived in the previous sections has been applied in a Computer program. We present three benchmark crack problems to demonstrate the accuracy and effectiveness of the proposed algorithm. The results are compared with those arrived at in the previous literature.

4.1 Fatigue Crack Growth in a Finite Plate

First, a benchmark problem of an edge-crack in a finite plate is considered. Using an implementation of the XFEM, the normalized stress intensity factors are computed and the crack propagation is investigated.

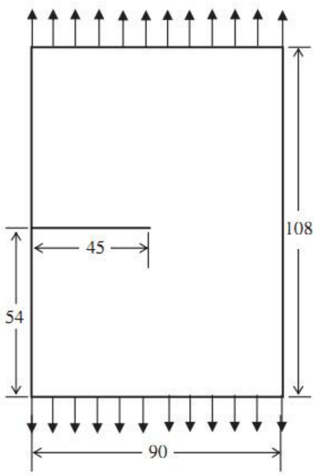

FIG. 5. EDGE-CRACK PROBLEM

Consider the edge-crack in a finite plate shown in fig. 5 with the dimensions of w = 90mm and h = 108 mm, thickness= 6mm, and under cyclic loading as those applied in [17], [18] (F_{max} = $16KN$, F_{min} = $8KN$). The parameters C = 7×10^{-8} and n=2.1 of Paris Low were considered for fatigue crack growth.

TABLE 1
COMPRESSION BETWEEN OUR RESULTS AND [17]

Our XFEM Results			Results of Ref. [17]	
a [mm]	Keq [Mpa(m)^0.5]	da/dN [*10^-5]	Keq [Mpa(m)^0.5]	da/dN [*10^-5]
45	16.2	0	15.9	0
47	17.8	2.9	17.5	2.6
49	19.6	3.6	19.2	3.5
51	21.6	4.4	21.2	4.3
53	24	5.5	23.5	5.3
55	26.7	6.9	26.2	6.7
57	29.8	8.7	29.2	8.4
59	33.3	11	32.8	11
61	37.7	14.3	37.1	14
63	42.8	18.7	42.1	18
65	48.9	24.8	48.3	24

Table 1 shows that the numerical results of XFEM are good agreement with the reference solution provided in [19].

4.2 Elastic stresses in flat punch

We present a benchmark contact problem to demonstrate the accuracy and effectiveness of the proposed contact algorithm. In order to exhibit the validity of the contact algorithm, a flat punch shown in fig. 6 with the dimensions h/w = 8/5, H/W=5/4 and w/W=5/8 is considered.

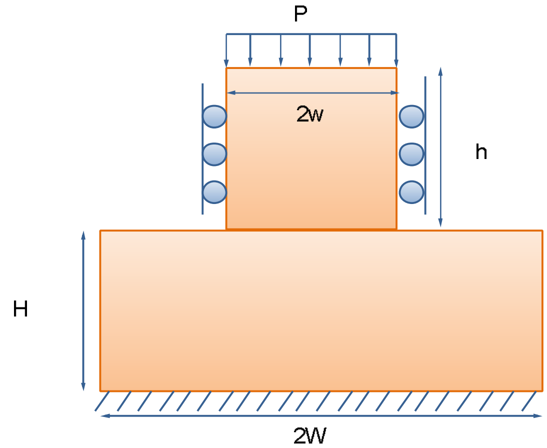

FIG. 6. GEOMETRY OF FLAT PUNCH

In this analysis, a material model with Young's modulus of E=200 Gpa and Poisson's ratio of ν=0.3, under the normal load of P=30 Mpa is supposed and the normal contact is implemented. Fig. 7 shows that the results are good agreement with the results of Eric and Urban [20].

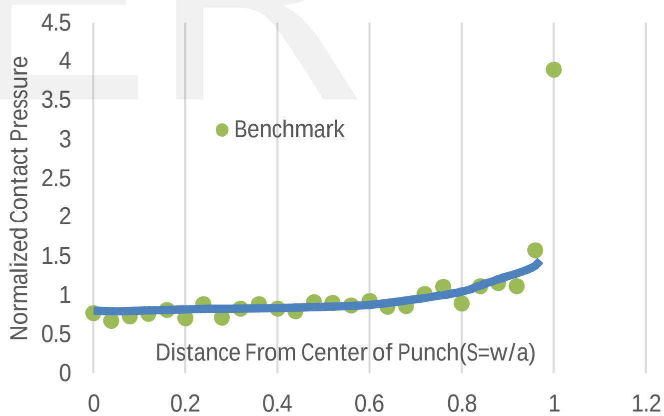

FIG. 7. NORMALIZED CONTACT PRESSURE VS. DISTANCE FROM CENTER OF PUNCH

4.3 A Fretting Fatigue Problem

A rectangular sample subjected to a variable bulk stress is shown in fig. 8. It has the dimensions of h = 2c = 2w = 10 mm and l = 40 mm. The material considered is aluminum alloy 7075- T6, with Young's modulus of E = 72 Gpa and Poisson's ratio of ν=0.3, same as those applied in [21], [22]. Those references solve this problem using ABAQUS Software. Plane strain condition and the complete sliding (Q=μP) with the friction coefficient of 0.8 are assumed. The normal load is constant, with a value of 60 M Pa and the bulk load is cyclical, with amplitude of ±110 Mpa.

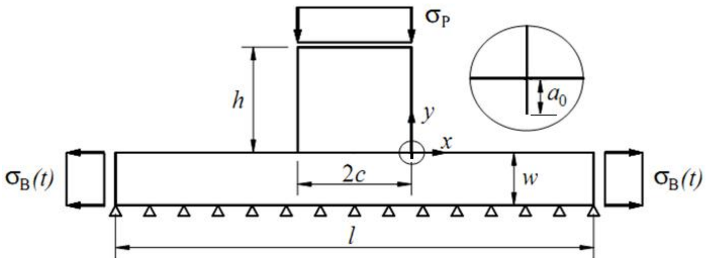

FIG. 8. GEOMETRY OF FRETTING FATIGUE PROBLEM

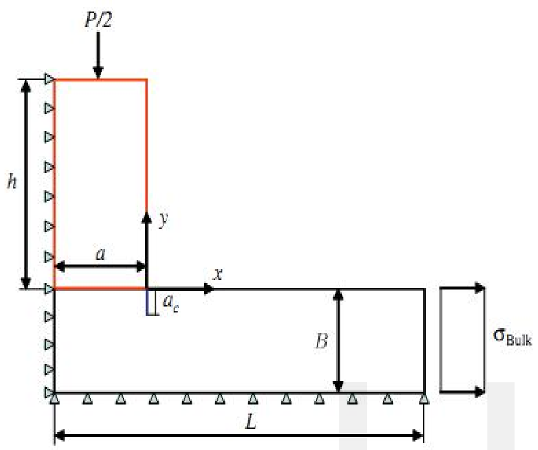

FIG. 9. AXISYMMETRIC MODEL OF FRETTING FATIGUE PROBLEM

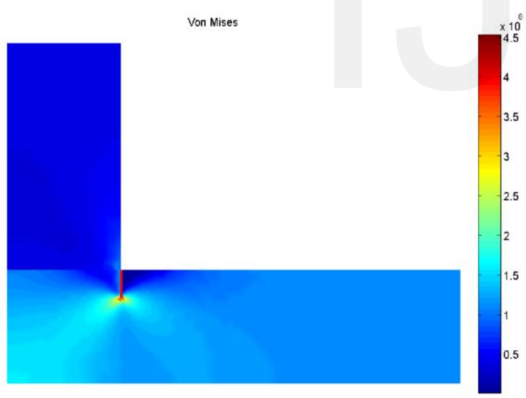

FIG. 10. VON MISES STRESS CONTOUR

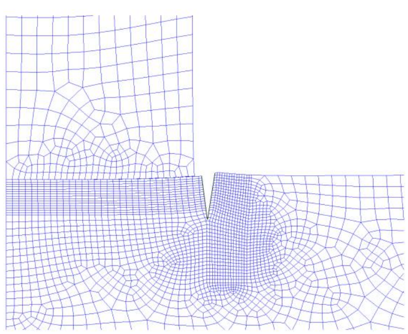

FIG. 11. CRACK OPENING (50X)

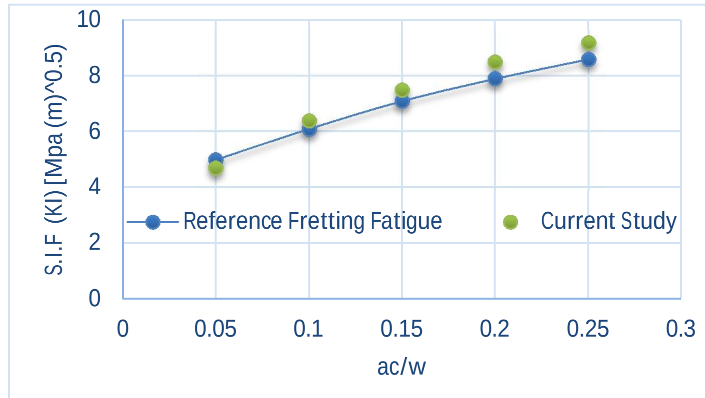

FIG. 12. STRESS INTENSITY FACTOR VS. VARIOUS NORMAL CRACK LENGTH

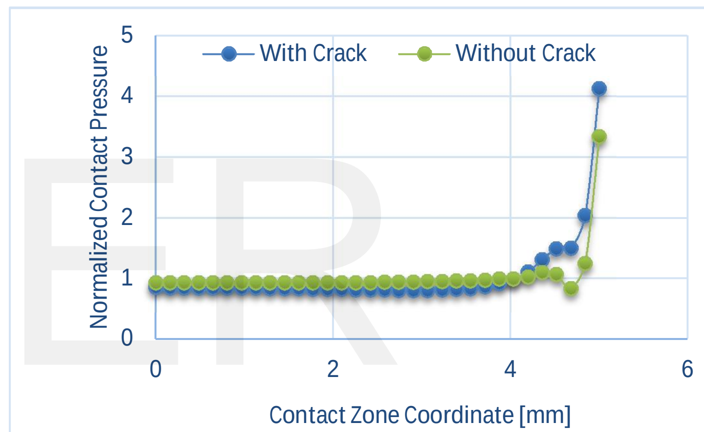

FIG. 13. NORMALIZED CONTACT PRESSURE

Fig.10 shows the von Mises' stresses at the Gauss quadrature points of the patch and one can notice that the singularity is captured. The deformations of two contact bodies and crack opening are shown in fig. 11. Results of this study are compared with results of sabsabi and al. [22]. Fig.12 shows that our results have good agreement with the experiment.
The results show that this algorithm can computes the correct contact stresses efficiently and is capable of investigating fretting fatigue problems.

5 CONCLUSION

The present paper developed an efficient hybrid technique that combined an extended finite element method and a mortar base frictional contact formulation. To demonstrate the capabilities of this code for solving contact problems, the numerical technique is applied to solve some typical problems of contact. In addition, a fretting fatigue problem is modeled and the results are validated by comparison with the results of fretting fatigue test. The results show that this algorithm can be utilized to predict the fretting fatigue life.

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